300 Years of Optimal Control: From The Brachystochrone to the Maximum Principle

Hector J. Sussmann and Jan C. Willems

Optimal control was born in 1697—300 years ago—in Groningen, a university town in the north of The Netherlands, when Johann Bernoulli, professor of mathematics at the local university from 1695 to 1705, published his solution of the *brachystochrone problem.* The year before he had challenged his contemporaries to solve this problem. We will tell the story of some of the events of 1696 and 1697—when solutions were submitted by Johann Bernoulli and such other giants as Newton, Leibniz, Tschirnhaus, l'Hopital, and Johann's brother, Jakob Bernoulli—and then sketch the evolution of this field until it reached maturity in our century. Since the birth of optimal control, like all births, did not take place in a vacuum, the historical context will first be described, by outlining briefly some of the main ideas and discoveries on curve minimization problems from classical Greece up to Bernoulli's time. We will then state the brachystochrone problem, present Bernoulli's solution, and also provide a short nontechnical interlude, dealing with Bernoulli's personality and with his exceptionally gifted family. Subsequently we will follow the intricate path that has led to the modern versions of the necessary conditions for a minimum, from the Euler-Lagrange equations to the work of Legendre and Weierstrass and, eventually, the maximum principle of optimal control theory. Finally, we will "close the loop" by returning to the brachystochrone from the perspective of modern optimal control.

Our thesis, that the brachystochrone marks the birth of optimal control, is undoubtedly somewhat controversial, and some readers—especially those who espouse views currently in vogue about the social construction of reality—might suspect that it is merely a reflection of the professional and nationalistic biases of the authors. We gladly plead guilty to most of this charge—and state for the record that *we are both control theorists, and one of us is a professor at Groningen*—asking only that the word "merely" be stricken out. Our biases may of course explain how

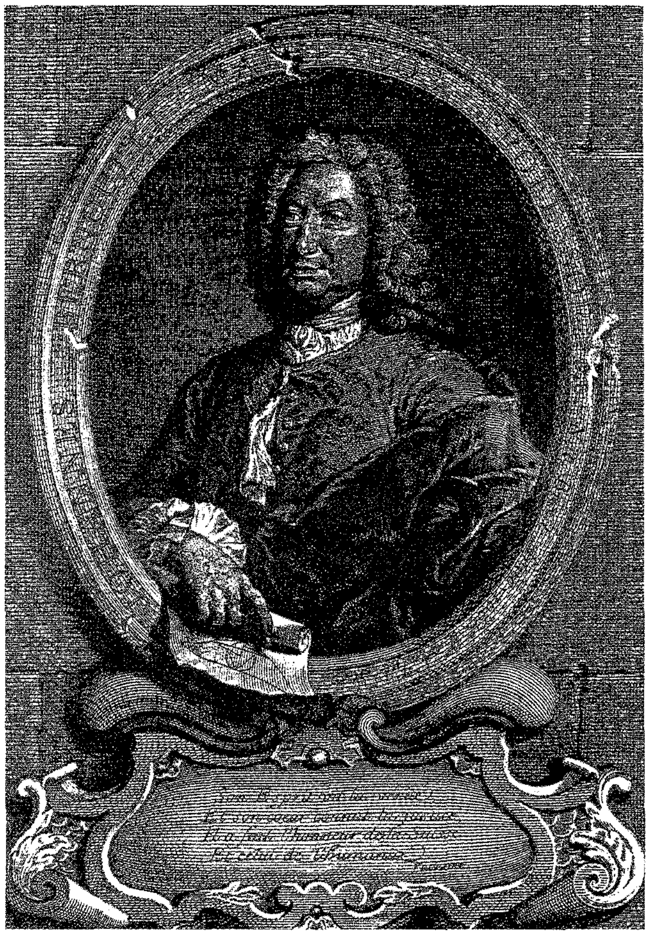

Fig. 1. Johann Bernoulli (1667-1748).

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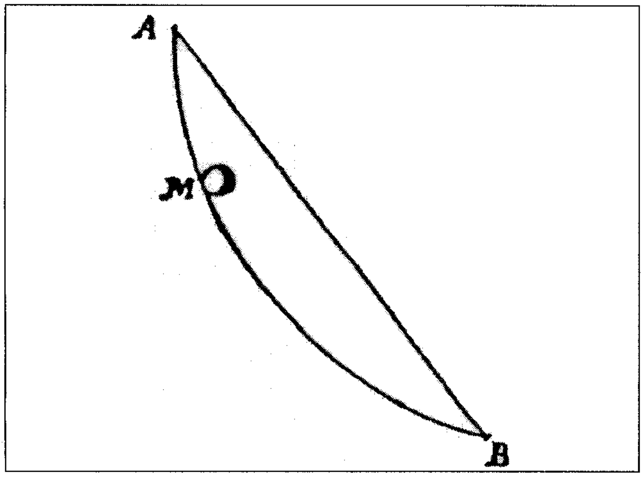

Fig. 2. The Brachystochrone Problem (Acta Eruditorum, June 1696, p. 269).

we became interested in this issue, but are not at all relevant to the merit and validity of our conclusion.

In this article, we will focus on point-to-point optimal control problems, where the objective is to transfer the state of a dynamical system with minimum cost from one point to another. This means that we are leaving out the whole area of transversality conditions, which arise when one considers "set-to-set" problems. Furthermore, we will not discuss at all the very important related question of sufficient conditions (and "Hamilton-Jacobi theory"), as well as the problem of finding optimal controllers, for example in the form of feedback laws, which is of course also a central concern of optimal control theory.

Bernoulli's Challenge

In the June 1696 issue of *Acta Eruditorum*, Bernoulli posed the following challenge (see Fig. 2):

Invitation to all mathematicians to solve a new problem. *If in a vertical plane two points A and B are given, then it is required to specify the orbit AMB of the movable point M, along which it, starting from A, and under the influence of its own weight, arrives at B in the shortest possible time. So that those who are keen of such matters will be tempted to solve this problem, is it good to know that it is not, as it may seem, purely speculative and without practical use. Rather it even appears, and this may be hard to believe, that it is very useful also for other branches of science than mechanics. In order to avoid a hasty conclusion, it should be remarked that the straight line is certainly the line of shortest distance between A and B, but it is not the one which is traveled in the shortest time. However, the curve AMB—which I shall divulge if by the end of this year nobody else has found it—is very well known among geometers.*

Later, at the suggestion of Leibniz, Bernoulli extended the deadline for the solution until Easter 1697, and on January 1, 1697, he published the announcement reproduced below, addressed to *The Sharpest Mathematical Minds of the Globe* (see Fig. 3).

Before 1696

Similar optimization problems had been studied at least since the Greeks. The oldest of all is the one of determining the shortest path joining two points, whose solution—which must have been well known since very ancient times—is a straight-line segment. Next came the *isoperimetric problem*, also known as *Dido's problem*, inspired by the mythical story told by Virgil (70-19 B.C.) in the *Aeneid* about the foundation of Carthage (*c.* 850 B.C.): the question is *to find the plane curve of a given length that encloses the largest possible area.* The solution was known by the Greeks to be the circle, although it took until the 19th century for this to be proved in a way that meets our contemporary standards of rigor.

Hero (or Heron) of Alexandria[1] showed in his *Catoptrics* that when a light ray emitted by an object is reflected by a mirror, it follows a path from the object to the eye which is the shortest of all possible such paths. In Hero's setting, which involved a single medium and therefore a constant speed of light, "shortest" was equivalent to "fastest." This was no longer the case in the work of Fermat (1601-1665), who formulated the general principle that *light rays follow the fastest—i.e., minimum time—paths.* This explained not only Hero's observation about reflection, but also Snellius' law of refraction. We shall see that Fermat's principle played a crucial role in Bernoulli's solution of the brachystochrone problem.

While all this was happening in the physics front, some progress was also made in the understanding of purely mathematical aspects of curve optimization problems. In particular, Newton had studied in 1685 the determination of the shape of a body with minimal drag, which was a true "calculus of variations" problem. But this remained an isolated piece of work which did not attract much attention and had no interesting spinoffs.

1696—1697: The Watershed

The events of 1696 and 1697 were a clear turning point. Bernoulli's 1696 challenge to his colleagues was taken up by the best mathematical minds of the time. Six mathematicians submitted solutions to the brachystochrone problem, and not just any six! Besides Johann's own solution, there was one by Leibniz, who called the problem *splendid* and solved it in a letter to Johann dated June 16, 1696; another one by Johann's elder brother Jakob; one by Tschirnhaus; one by l'Hopital, and, finally, one by Newton. Newton's solution was presented to the Royal Society on February 24, 1697, and published, anonymously and without proof, in the *Philosophical Transactions.* However, the identity of the author was clear to Bernoulli, since, as he noted, *ex ungue leonem* (you can tell the lion by its claws). Johann's solution was published in the *Acta Eruditorum* of May 1697, almost exactly 300 years before this magazine article, and the same issue also contained Jakob's solution, reprinted Newton's anonymous solution, and included the contributions by Tschirnhaus and l'Hopital, as well as a short note by Leibniz, remarking that he would not reproduce his own solution, since it was similar to that of Bernoulli. He also noted who else, in his opinion, could solve the problem: *l'Hopital, Huygens, were he alive, Hudde, if he had not given up mathematics,*[2] *and Newton, if he would take the trouble.*

The solutions of Bernoulli's problem were as beautiful as could have been expected given the eminence of the personalities who took up his challenge and found the correct answer. Moreover, this work was followed by a period of intense activity on problems of a similar kind, whose origin is directly traceable to the events of 1696-1697, and in many cases specifically to the Bernoullis, both intellectually and in terms of personal contacts. For example, Euler was a student of Bernoulli in Basel, and Lagrange became interested in variational problems by reading Euler's works. From this research, general techniques eventually emerged in the work of Euler and Lagrange. So there is no doubt that something important in the history of mathematics happened in 1696-1697. For example, D.J. Struik, in [9], p. 392, says of the articles published in the May 1697 *Acta Eruditorum* that "these papers opened the history of a new field, the calculus of variations."

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[1] Exact dates unknown. Believed by historians to have flourished about 100 B.C., although some attribute his optics work to a "Hero the Younger," who may have lived in the 7th or 8th century A.D.

[2] Hudde became mayor of Amsterdam, and Huygens died in 1695.

Why Optimal Control?

The conventional wisdom holds that optimal control theory was born about 40 years ago in the former Soviet Union, with the work on the "Pontryagin maximum principle" by L.S. Pontryagin and his group (cf. [8]). Some mathematicians believe that this new theory was no more than a minor addition to the classical calculus of variations, essentially involving the incorporation of inequality constraints. The article by L. Markus in [6] describes the unenthusiastic reaction at the 1958 International Congress of Mathematicians to the announcement of the maximum principle by the Soviet group. In addition, it is likely that other, nonmathematical, factors may also have contributed to the negative reaction. Among these, two reasons clearly stand out: first of all Pontryagin's personality and, in particular, his notorious anti-Semitism, and second, the feeling that many held that the result was primarily intended for military applications.

We believe that optimal control is significantly richer and broader than the calculus of variations, from which it differs in some fundamental ways, as we now explain.

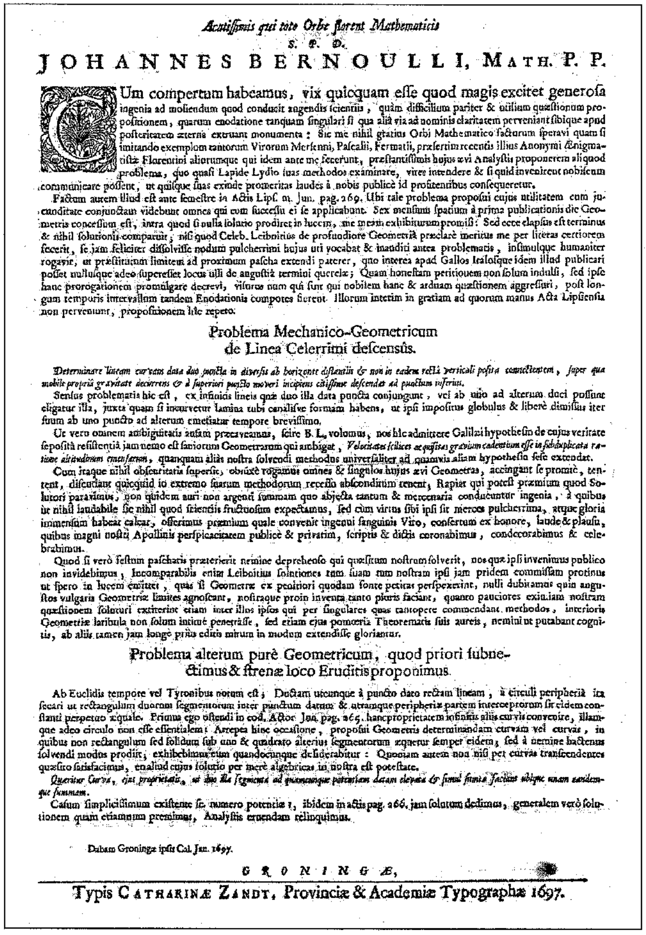
The calculus of variations deals mainly with optimization problems of the following "standard" form[3]:

$$\begin{aligned}&\text{minimize } I = \int_a^b L(q(t), \dot{q}(t), t)\,dt,\\ &\text{subject to } q(a) = \bar{q} \text{ and } q(b) = \hat{q}\,,\end{aligned} \qquad (1)$$

or, equivalently, of the form

$$\begin{aligned}&\text{minimize } I = \int_a^b L(q(t), u(t), t)\,dt,\\ &\text{subject to } q(a) = \bar{q},\ q(b) = \hat{q}\,, \text{ and } \dot{q}(t) = u(t)\\ &\text{for } a \le t \le b\,.\end{aligned} \qquad (2)$$

The distinctive feature of these problems is that the minimization of (1) takes place in the space of "all" curves, so nothing interesting happens on the level of the set of curves under consideration, and all the nontrivial features of the problem arise because of the Lagrangian L.



Acutiſſimis qui toto Orbe florent Mathematicis
S. P. D.
JOHANNES BERNOULLI, Math. P. P.

CUm compertum habeamus, vix quicquam eſſe quod magis excitet generoſa ingenia ad moliendum quod conducit augendis ſcientiis, quàm difficilium pariter & utilium quæſtionum propoſitionem, quarum enodatione tanquam ſingulari ſi qua aliâ via ad nominis claritatem perveniant ſibique apud poſteritatem æterna extruant monumenta: Sic me nihil gratius Orbi Mathematico facturum ſperavi quam ſi imitando exemplum tantorum Virorum Merſenni, Paſcalii, Fermatii, præſertim recentis illius Anonymi Ænigmatiſtæ Florentini aliorumque qui idem ante me fecerunt, præſtantiſſimis hujus ævi Analyſtis proponerem aliquod problema, quo quaſi Lapide Lydio ſuas methodos examinare, vires intendere & ſi quid invenirent nobiſcum communicare poſſent, ut quiſque ſuas exinde promeritas laudes à nobis publicè id profitentibus conſequeretur.

Factum autem illud eſt ante ſemeſtre in Actis Lipſ. m. Jun. pag. 269. Ubi tale problema propoſui cujus utilitatem cum jucunditate conjunctam videbunt omnes qui cum ſucceſſu ei ſe applicabunt. Sex menſium ſpatium à prima publicationis die Geometris conceſſum eſt, intra quod ſi nulla ſolutio prodiret in lucem, me meam exhibiturum promiſi: Sed ecce elapſus eſt terminus & nihil ſolutionis comparuit; niſi quod Celeb. Leibnitius de profundiore Geometriâ præclarè meritus me per literas certiorem fecerit, ſe jam feliciter diſſolviſſe nodum pulcherrimi hujus uti vocabat & inauditi antea problematis, inſimulque humaniter rogavit, ut præſtitutum limitem ad proximum paſcha extendi paterer, quo interea apud Gallos Italoſque idem illud publicari poſſet nulluſque adeo ſupereſſet locus ulli de anguſtiâ termini querelæ; Quam honeſtam petitionem non ſolum indulſi, ſed ipſe hanc prorogationem promulgare decrevi, viſurus num qui ſint qui nobilem hanc & arduam quæſtionem aggreſſuri, poſt longum temporis intervallum tandem Enodationis compotes fierent. Illorum interim in gratiam ad quorum manus Acta Lipſienſia non perveniunt, propoſitionem hîc repeto.

Problema Mechanico-Geometricum
de Linea Celerrimi deſcenſûs.

Determinare lineam curvam data duo puncta in diverſis ab horizonte diſtantiis & non in eadem rectâ verticali poſita connectentem, ſuper qua mobile propriâ gravitate decurrens & à ſuperiori puncto moveri incipiens citiſſime deſcendat ad punctum inferius.

Senſus problematis hic eſt, ex infinitis lineis quæ duo illa data puncta conjungunt, vel ab uno ad alterum duci poſſunt eligatur illa, juxta quam ſi incurvetur lamina tubi canaliſve formam habens, ut ipſi impoſitus globulus & liberè dimiſſus iter ſuum ab uno puncto ad alterum emetiatur tempore breviſſimo.

Ut vero omnem ambiguitatis anſam præcaveamus, ſcire B. L. volumus, nos hic admittere Galilæi hypotheſin de cujus veritate ſepoſitâ reſiſtentiâ jam nemo eſt ſaniorum Geometrarum qui ambigat, *Velocitates ſcilicet acquiſitas gravium cadentium eſſe in ſubduplicata ratione altitudinum emenſarum*, quanquam aliàs noſtra ſolvendi methodus univerſaliter ad quamvis aliam hypotheſin ſeſe extendat.

Cum itaque nihil obſcuritatis ſuperſit, obnixè rogamus omnes & ſingulos hujus ævi Geometras, accingant ſe promtè, tentent, diſcutiant quicquid in extremo ſuarum methodorum receſſu abſconditum tenent; Rapiat qui poteſt præmium quod Solutori paravimus, non quidem auri non argenti ſummam quo abjecta tantum & mercenaria conducuntur ingenia, à quibus ut nihil laudabile ſic nihil quod ſcientiis fructuoſum expectamus, ſed cùm virtus ſibi ipſi ſit merces pulcherrima, atque gloria immenſum habeat calcar, offerimus præmium quale convenit ingenui ſanguinis Viro, conſertum ex honore, laude & plauſu, quibus magni noſtri Apollinis perſpicacitatem publicè & privatim, ſcriptis & dictis coronabimus, condecorabimus & celebrabimus.

Quod ſi verò feſtum paſchatis præterierit nemine deprehenſo qui quæſitum noſtrum ſolverit, nos quæ ipſi invenimus publico non invidebimus; Incomparabilis enim Leibnitius ſolutiones tum ſuam tum noſtram ipſi jam pridem commiſſam protinus ut ſpero in lucem emittet, quas ſi Geometræ ex penitiori quodam fonte petitas perſpexerint, nulli dubitamus quin anguſtos vulgaris Geometriæ limites agnoſcant, noſtraque proin inventa tanto pluris faciant, quanto pauciores eximiam noſtram quæſtionem ſoluturi extiterint etiam inter illos ipſos qui per ſingulares quas tantopere commendant methodos, interioris Geometriæ latibula non ſolum intimè penetraſſe, ſed etiam ejus pomœria Theorematis ſuis aureis, nemini ut putabant cognitis, ab aliis tamen jam longè priùs editis mirum in modum extendiſſe gloriantur.

Problema alterum purè Geometricum, quod priori ſubnectimus & ſtrenæ loco Eruditis proponimus.

Ab Euclidis tempore vel Tyronibus notum eſt; Ductam utcunque à puncto dato rectam lineam, à circuli peripheriâ ita ſecari ut rectangulum duorum ſegmentorum inter punctum datum & utramque peripheriæ partem interceptorum ſit eidem conſtanti perpetuo æquale. Primus ego oſtendi in eod. Actor. Jun. pag. 265. hanc proprietatem infinitis aliis curvis convenire, illamque adeo circulo non eſſe eſſentialem: Arrepta hinc occaſione, propoſui Geometris determinandam curvam vel curvas, in quibus non rectangulum ſed ſolidum ſub uno & quadrato alterius ſegmentorum æquetur ſemper eidem; ſed à nemine hactenus ſolvendi modus prodiit; exhibebimus eum quandocunque deſiderabitur: Quoniam autem non niſi per curvas tranſcendentes quæſito ſatisfacimus, en aliud cujus ſolutio per merè algebricas in noſtra eſt poteſtate.

Quæritur Curva, ejus proprietatis, ut duo illa ſegmenta ad quamcunque potentiam datam elevata & ſimul ſumta faciant ubique unam eandemque ſummam.

Caſum ſimpliciſſimum exiſtente ſc. numero potentiæ 1, ibidem in actis pag. 266. jam ſolutum dedimus, generalem verò ſolutionem quam etiamnum premimus, Analyſtis eruendam relinquimus.

Dabam Groningæ ipſis Cal. Jan. 1697.

GRONINGÆ,
Typis Catharinæ Zandt, Provinciæ & Academiæ Typographæ 1697.

Fig. 3. Johann Bernoulli's announcement.

Optimal control problems, by contrast, involve a minimization over a set $\mathcal{C}$ of curves which is itself determined by some dynamical constraints. For example, $\mathcal{C}$ might be the set of all curves $t \mapsto q(t)$ that satisfy a differential equation

$$\dot{q}(t) = f(q(t), u(t), t) \qquad (3)$$

for some choice of the "control function" $t \mapsto u(t)$. Even more precisely, since it may happen that a member of $\mathcal{C}$ does not uniquely determine the control u that generates it, we should be talking *about trajectory-control pairs* $(q(\cdot), u(\cdot))$. So in an optimal control problem there are at least two objects that give the situation interesting structure, namely, the dynamics f and the functional I to be minimized. In particular, optimal control theory contains, at the opposite extreme from the calculus of variations, problems where the "Lagrangian" L is $\equiv 1$, i.e. completely trivial, and therefore all the interesting action occurs because of the dynamics f. Such problems, in which it is desired to minimize time—i.e., the integral I of (2) with $L \equiv 1$—among all curves $t \mapsto q(t)$ that satisfy endpoint constraints as in (2) and are solutions of (3) for some control $t \mapsto u(t)$, are called *minimum time problems*. It is in these problems that the difference between optimal control and the calculus of variations is most clearly seen, and it is no accident that these were the prob-

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[3] In what follows, we will discuss the work of several authors from the 17th to the 19th centuries. In the interests of clarity and consistency, we will always use our own notations and mathematical terminology rather than those of the authors under discussion. So, for example, the letter L will always stand for the "Lagrangian," the state variables will usually—but not always—be called q, and the independent variable—often called x or y in early papers on the subject—will usually be t, and should be thought of as time. We will use dots—and on a few occasions primes, and also d/dt, when we want to differentiate a long expression—to denote differentiation with respect to time (cf. Equation (12) below for an example of the use of these notations).

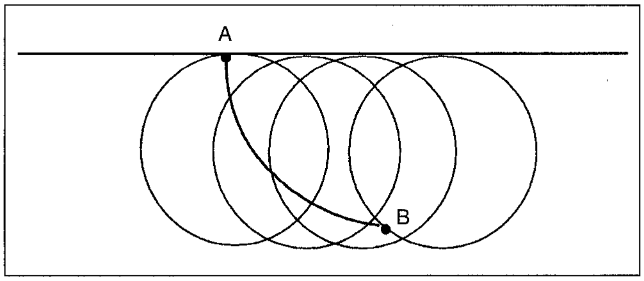

Fig. 4. The Brachystochrone cycloid (Acta Eruditorum, May 1697).

lems that propelled the development of optimal control in the early 1960s, and that time-optimal control is prominently represented in today's research and in modern optimal control textbooks.

Within this framework, we can state the first of our reasons for claiming that the brachystochrone problem marks the birth of optimal control: *Bernoulli's problem, as posed in the Acta Eruditorum, is a true minimum time problem of the kind that is studied today in optimal control theory.* Bernoulli called the fastest path the *brachystochrone* (from the Greek words βραχιστος: shortest, and χρονος: time). Moreover, *the brachystochrone problem is the first one ever to deal with a dynamical behavior and explicitly ask for the optimal selection of a path.* In both the isoperimetric problem and Newton's minimal drag problem the curves to be computed are not thought of as paths of a moving body or particle. Finally, and most importantly, *a large part of the subsequent history of the calculus of variations can be best understood as the search for the simplest and most general statement of the necessary conditions for optimality, and this statement is provided by the maximum principle of optimal control theory.*

The above reasons are, in our view, compelling arguments in favor of our claim that 1696 deserves to be called the year of the birth of optimal control.

Bernoulli's Solution of the Brachystochrone Problem

We start by describing Johann's Bernoulli's solution.[4]

Let us first formulate the brachystochrone problem in modern mathematical language. Choose x and y axes in the plane with the y axis pointing downwards. Use $(0,0)$ and (a,b) to denote, respectively, the coordinates of the end points A and B. A path $f: [0,T] \rightarrow \mathbb{R}^2$, defined on an interval $[0,T]$, and having components $f_1(t)$, $f_2(t)$, is said to be a *feasible trajectory* (or *feasible path*) if

(i) $f(0) = (0,0)$, $f(T) = (a,b)$, and f is Lipschitz continuous,

(ii) $\frac{1}{2}\left(\left|\dot{f}_1(t)\right|^2 + \left|\dot{f}_2(t)\right|^2\right) = gf_2(t)$ for almost all $t \in [0,T]$.

Here g is the gravitational constant. Condition (i) states that the path f must start at A and end at B. Condition (ii) reflects conservation of energy: at each instant t, the kinetic energy of the body must equal the decrease of potential energy due to its loss of height. (The law that a body which has fallen from a height h has velocity proportional to $\sqrt{h}$ was due to Galileo, and was well known in Bernoulli's time.)

A feasible path $f^* : [0,T^*] \rightarrow \mathbb{R}^2$ is said to be *optimal* if there exists no feasible path $f: [0,T] \rightarrow \mathbb{R}^2$ for which $T < T^*$. A *brachystochrone* is a curve in $\mathbb{R}^2$ traversed by an optimal feasible path, i.e., a subset B of $\mathbb{R}^2$ of the form $B = (x,y) \in \mathbb{R}^2$: there exists $t \in [0,T^*]$, such that $(x,y) = f^*(t)\}$ where $f^* : [0,T^*] \rightarrow \mathbb{R}^2$ is an optimal feasible path.

One obvious fact is that *the solution cannot always be a straight line*, a possibility that Bernoulli rightly warns against. For example, consider the extreme case when $b = 0$. It is easy to see that it takes finite time to roll from A to B on a half circle, since it will take finite time to roll from A to the bottom of the circle, and the same time to climb back up to B. Since, however, the straight-line segment from A to B is horizontal, the speed of motion along it vanishes. So, the straight line segment cannot be an optimal path, because the motion along it takes infinite time.

It turns out that the brachystochrone is a cycloid. It is the curve described by a point P in a circle that rolls without slipping on on the x axis, in such a way that P passes through A and then through B, without hitting the x axis in between. It is easy to see that this defines the cycloid uniquely (see Fig. 4).

Bernoulli's ingenious derivation of the brachystochrone has been the subject of numerous accounts, but since this event plays a crucial role in our own story, we will outline the proof again.

Bernoulli based his derivation on Fermat's minimum time principle. If we imagine for a moment that instead of dealing with the motion of a moving body we are dealing with a light ray, condition (ii) above gives us a formula for the "speed of light" c as a function of position: $c = \sqrt{2gy}$. Let us rescale—or, if the reader so prefers, "change our choice of physical units"—so that $2g = 1$. Then our problem is exactly equivalent to that of determining the light rays—i.e., the minimum-time paths—in a plane medium where the speed of light c varies continuously as a function of position according to the formula $c = \sqrt{y}$.

It is at least intuitively clear that, if we discretize our problem by dividing the half-plane into horizontal strips $S_k = \{(x,y) : y_k \le y \le y_{k+1}\}$ of height δ, for $k = 0, 1, \ldots$, where $y_k = kd$, and treating c in each strip S_k as a constant c_k (by, say, setting $c_k = \sqrt{y_{k+1}}$), then the light rays for the discretized problem should approach those for the original problem as $\delta \downarrow 0$. The light rays of the discretized problem can be studied using the law of refraction of light. Clearly, the paths will be straight-line segments within each individual strip, and all that needs to be done is to determine how these rays bend as they cross the boundary between two strips. The answer is provided by the laws of optics as developed by Snellius, Fermat, and Huygens.

Snellius had observed that, if two media are separated by a straight line, and a light ray is refracted at the boundary between them, then the ratio of the sines of the incidence angles between the light rays and the normal to the boundary is constant. Fermat subsequently showed that this is precisely what happens when light is assumed to follow a minimum-time path. Applying this to the situation of the two media separated by a horizontal leads to the following optimization problem. Assume that we have two points, the first, P_1, located above, and the second, P_2, lying below the boundary. Suppose a light ray travels with speed v_1 in the medium above the horizontal line and with speed v_2 in the medium below the line. Of course, when $v_1 = v_2$, this fastest path is the straight line from P_1 to P_2. This implies that the fastest path to travel from P_1 to P_2, when $v_1 \neq v_2$ is a broken line consisting of a straight line from P_1 to some point P' on the boundary, and another straight line from P' to P_2. The problem is thus reduced to finding the point P'. This, is, however, a simple calculus question, and it turns out that the point P' is determined by the equation $\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2}$ or, equivalently, $\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}$.

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[4] Jakob's solution was quite different from Johann's, and at first sight seemed clumsier, but in the long run it has turned out to be more akin to the mainstream ideas of the calculus of variations, Hamilton-Jacobi theory, and dynamic programming, and is therefore widely considered to be of great historical importance in the development of optimal control. It will not, however, be discussed here, due to lack of space. Goldstine's book [7] gives an excellent account.

This law relating the incidence angles to the velocities of propagation is due to Huygens, and implies the law of Snellius. Bernoulli used Huygens' law to conclude that the quantity $\frac{\sin\theta_k}{\sqrt{y_{k+1}}}$ will be a constant, since in each strip S_k the speed of our light ray is $\sqrt{y_{k+1}}$. Passing to the limit as $\delta \downarrow 0$, we conclude that the sine of the angle θ between the tangent to the brachystochrone and the vertical axis must be proportional to $\sqrt{y}$. Since $\sin\theta = \frac{dx}{\sqrt{dx^2 + dy^2}}$, we find that $\frac{dx^2}{dx^2 + dy^2} = Ky$, where K is a constant. Then $\frac{dx^2 + dy^2}{dx^2} = \frac{1}{Ky}$, i.e., $1 + y'(x)^2 = \frac{C}{y}$, where $C = \frac{1}{K}$. So the curve described by expressing the y-coordinate of the brachystochrone as a function of its x-coordinate will satisfy the differential equation

$$y'(x) = \sqrt{\frac{C - y(x)}{y(x)}}\,, \tag{4}$$

with C a constant. The curves given by the parametric equations

$$\begin{aligned} x(\varphi) &= x_0 + \frac{C}{2}(\varphi - \sin\varphi)\,, \\ y(\varphi) &= \frac{C}{2}(1 - \cos\varphi)\,, \\ 0 &\leq \varphi \leq 2\pi\,, \end{aligned} \tag{5}$$

satisfy (4). It is easily seen that these equations specify the cycloid generated by a point P on a circle of diameter C that rolls without slipping on the horizontal axis, in such a way that P is at $(x_0, 0)$ when $\varphi = 0$.

The argument that we have presented is Bernoulli's, and Equation (4) appears in his paper, followed by the statement "from which I conclude that the *Brachystochrone* is the ordinary *Cycloid*." (He actually wrote $dy = dx\sqrt{\frac{x}{a-x}}$, but he was using x for the vertical coordinate and y for the horizontal one. Cf. [9], p. 394).

In contemporary mathematics, the symbol $\sqrt{r}$ usually stands for the *nonnegative* square root of r, but it is obvious that Johann Bernoulli did not have this in mind. What he meant was, clearly, what we would write as

$$y'(x) = \pm\sqrt{\frac{C - y(x)}{y(x)}} \tag{6}$$

or, equivalently,

$$y(x)(1 + y'(x)^2) = \text{constant}\,. \tag{7}$$

In particular, the solution curves should be allowed to have a negative slope. But y' should stay continuous, so that a switching from a + to a - solution of (6) is not permitted.

Even with the more accurate rewriting (7), the differential equation derived by Bernoulli also has *spurious* solutions, not given by (5)! Indeed, for any $\bar{y} > 0$, the constant function $y(x) = \bar{y}$ is a solution, corresponding to $C = \bar{y}$. More generally, one can take an ordinary cycloid given by (5), follow it up to $\varphi = \pi$—so that $dy/dx = 0$—then follow the constant solution $y(x) = C$ for an arbitrary time T, and then continue with a cycloid given by (5). Such paths are, indeed, compatible with Huygens' law of refraction.

It is easily understood that the laws of Snellius and Huygens cannot explain why a light ray has to bend upward or downwards once it is horizontal. As such Bernoulli's argument is certainly incomplete when the brachystochrone cycloid connecting A and B first bottoms out before climbing back up to the point B. There is no reason why it should not proceed horizontally once it has reached the lowest point. This shortcoming in Bernoulli's argument seems to have escaped historians. We shall later see that the maximum principle does exclude these horizontal motions.

The spurious solutions, and all the other problems, such as the apparent arbitrariness of the requirement that y' be continuous, can be eliminated in a number of ways. For example, one can prove directly that the spurious trajectories are not optimal, or one can use, as an alternative to Bernoulli's method, the calculus of variations approach, based on the Euler-Lagrange equation (10) below.

It is easy to see that the brachystochrone problem can be put in the "standard" form (1), provided we *postulate*[5] that it suffices to consider curves in the x,y plane that are graphs of functions $y = y(x)$ defined on $[0,a]$. Then the dynamical constraint (ii)—with $2g = 1$, as before—becomes $dx^2 + dy^2 = y\,dt^2$, which gives $dt = \frac{\sqrt{dx^2 + dy^2}}{\sqrt{y}} = L(y, \dot{y})dx$, where

$$L(y,u) = y^{-1/2}\,(1 + u^2)^{1/2} \tag{8}$$

and we are using x rather than t for the time variable, and writing $\dot{y}$ for dy/dx. So Bernoulli's problem becomes that of minimizing the integral $\int_0^a L(y(x), \dot{y}(x))dx$ subject to $y(0) = 0$ and $y(a) = b$.

This gives the Euler-Lagrange equation

$$1 + y'(x)^2 + 2y(x)y''(x) = 0\,, \tag{9}$$

which is *stronger* than (7), since (7) is equivalent to $y' + y'^3 + 2yy'y'' = 0$, i.e., to $y'(1 + y'^2 + 2yy'') = 0$, whose solutions are those of (9) plus the spurious solutions found earlier. It is easy to see that the solutions of the Euler-Lagrange equation (9) are *exactly* the curves given by (5), without any extra spurious solutions, showing that, for the brachystochrone problem, *the Euler-Lagrange method gives better results than Bernoulli's ap-*

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[5] With optimal control, this "postulate" becomes a *provable conclusion*, cf. "Finale for Brachystochrone and Control" below.

proach. (We will see in a later section that *optimal control is even better.*)

Bernoulli was originally under the mistaken impression that the brachystochrone problem was new. However, Leibniz knew better: in 1638 Galileo, in his book on the *Two New Sciences,* had formulated the brachystochrone problem and even suggested a solution: he seems to have thought it was a circle. Galileo had actually shown—correctly—that an arc of a circle always did better than a straight line—except, of course, when $a = 0$.

Bernoulli considered the fact that Galileo had been mistaken on two counts, by thinking that the catenary was a parabola, and that the brachystochrone was a circle, as conclusive evidence of the superiority of differential calculus (or the *Nova Methodus* as they called it).

He was thrilled by his discovery that the brachystochrone was a cycloid. This curve had been introduced by Galileo, who had given it its name: *related to the circle.* Huygens had discovered a remarkable property of the cycloid: it is the only curve such that a body falling under its own weight is guided by this curve so as to oscillate with a period that is independent of the initial point where the body is released. Contrary to what Galileo thought, the circle has this property only approximately: the period of oscillation of a pendulum is a function of its amplitude. Therefore, Huygens called this curve, the cycloid, the *tautochrone* (from ταντος: equal, and χρονος: time). Bernoulli was amazed and somewhat puzzled, it seems, by the coincidence that the cycloid turns out to be both the brachystochrone and the tautochrone, so that two rather different properties related to the time traveled on it by a body falling under its own weight led, in the end, to the same curve. He concluded that *nature always arranges things in the simplest manner, as here, by giving the same curve two different properties.*

Johann Bernoulli and his Family

We now sketch some of the historical context surrounding the life and work of Bernoulli. The Bernoullis were a Protestant family originally from Antwerp in Flanders. They fled Antwerp in 1583 to escape the religious oppression of the Spanish rulers and, after spending some time in Frankfurt, finally settled in Basel, Switzerland, early in the 17th century. Among its members there were eight mathematicians in three consecutive generations. Most of them ended up as professors in Basel, but many spent extensive periods in other universities in Europe. The most prominent of the Bernoullis were Jakob (1654-1705), his younger brother Johann (1667-1748), the protagonist of our story, and Johann's son, Daniel (1700-1782), born in Groningen while his father was a professor there. Jakob Bernoulli made important contributions, in particular, to probability theory. (Bernoulli distributions are named after him.) Daniel is the discoverer of Bernoulli's law in hydrodynamics, one of the great laws in physics.

At the time that Bernoulli came of age, mathematics was going through a revolution. In 1684, Leibniz published his first article about differential calculus in the *Acta Eruditorum.* This article was entitled *Nova methodus pro maximis et minimis, itemque tangentibus, quae nec fractas, nec irrationales quantitas moratur, & singulare pro illis calculi genus.* He showed the power of the *Nova Methodus* by finding maxima and minima for a number of examples much more effectively than had been possible before. Johann and Jakob Bernoulli were among the first to master Leibniz' technique, and, in 1691, Johann achieved his first success by using the differential calculus to determine the catenary, the shape of a hanging chain. In his mere mid-20's, Johann was hired by the Marquis de l'Hopital, a French nobleman and one of the leading mathematicians of his time, to teach him the differential calculus. While he received a handsome payment for his services, he was bound by contract to let the Marquis take credit for the discoveries made by Johann during this teaching. Johann always claimed that he was the true discoverer of l'Hopital's rule about the limit of $\frac{0}{0}$, which appeared in the Marquis' book, *Analyse des Infiniment Petits.* His contemporaries tended to ignore this claim, since Johann was not known to be particularly generous to others or objective about his own achievements. However, in 1922, the original notes of these lectures were discovered, which brought positive evidence for Johann's claim.

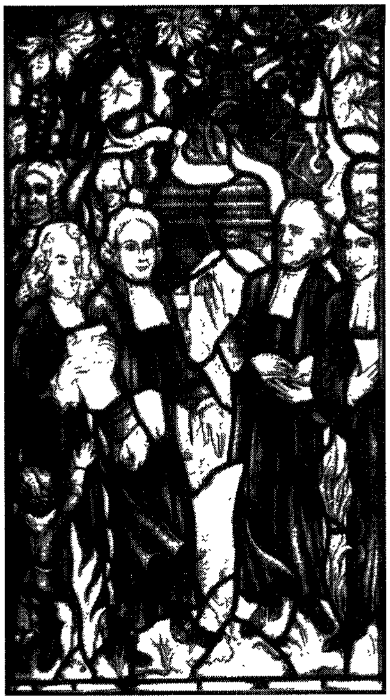
Fig. 5. Johann and Daniel Bernoulli.

Johann Bernoulli was not an easy person. He often quarreled openly with his colleagues, and complained about his salary, his health, his work. In 1695, shortly after taking up the chair in Groningen that had been offered to him on the recommendation of Huygens, he vented his disenchantment in a letter to Leibniz, who had encouraged him to accept the offer: *I have not met any of the practitioners of Algebra, which you consider present in Holland. To the contrary, I have not had the honor of meeting a single person who would even deserve to be called a "mediocre mathematician."* In the same letter he complained that his teaching took too much of his time, and that *the more progress the students make, the less progress I make.* Bernoulli expressed such politically incorrect views not only in private letters, but also publicly. While in Groningen he got into serious difficulties with the local protestant theologians and clergy, who disapproved of the way new discoveries in the physical sciences cast doubt on the validity of revealed truth.

In his disputes with his mathematical colleagues he was unrelenting. He was perhaps the most abrasive contender in the bitter controversy between the English, Newtonian, and the continental, Leibnizian, schools, regarding the originality and rigor of the differential calculus. He "was a man of violent likes and dislikes: Leibniz and Euler were his gods; Newton he positively hated and greatly underestimated." ([1], p. 135.) His rivalry with his brother Jakob became an embarrassment to the scientific community, and when in 1699 they were both elected to the Paris Academy, it was on the explicit condition that they promise to cease arguing, a promise that of course was not kept. Even more peculiar was Johann's rivalry with his own son Daniel, whom he criticized—for being a Newtonian—and plagiarized—on the law of hydrodynamics—and of whose success he was allegedly very jealous. Johann once threw Daniel out of the house for having won a French Academy of Sciences prize for which Johann had also been a candidate, cf. [1], p. 134. Daniel, however, remained dutifully respectful towards his father, but frequently expressed his misgivings to his friend Euler (a student of Johann in Basel and a colleague of Daniel in Saint Petersburg).

Fig. 5 is a photograph of a stained glass window of the Academy Building (the main venue of the university) in Groningen. It shows Daniel Bernoulli sweetly clutching his father's robe, while Johann shows off his brachystochrone.

At the occasion of the 300th anniversary of the appointment of Bernoulli and the discovery of the brachystochrone, the University of Groningen erected the monument shown in Fig. 6. It consists of an artist's rendering of the brachystochrone, with the circle that generates the cycloid. In the background, one can see the building of the mathematics department, where the second author of this article has his office.

Euler, Lagrange, Legendre

With the work of Johann and Jakob Bernoulli, Leibniz, Tschirnhaus, Newton, and l'Hopital on the brachystochrone, optimal control got off to a spectacular start. Let us now look at some critical events in its later evolution.

The next chapter of our tale is the work of Euler (1707-1793) and Lagrange (1736-1813). Leonhard Euler entered the University of Basel at the age of 13, and became a student of Bernoulli, who gave him private lessons once a week. In Basel, he worked on isoperimetric problems in 1732 and 1736. In 1744 he published his book *The Method of Finding Plane Curves that Show Some Property of Maximum or Minimum*, where he gave a general procedure for writing down what became known as Euler's equation.

And then Lagrange entered the stage. In H. Goldstine's words ([7], p. 110.):

On 12 August 1755 a 19-year-old Ludovico de la Grange Tournier of Turin, wrote Euler a brief letter to which was attached an appendix containing mathematical details of a very beautiful and revolutionary idea. He saw how to eliminate from Euler's methods the tedium and need for geometrical insight and to reduce the entire process to a quite analytic machine or apparatus, which could turn out the necessary condition of Euler and more, almost automatically. This basic idea of Lagrange ushered in a new epoch in the calculus of variations. Indeed, after seeing Lagrange's work, Euler dropped his own method, espoused that of Lagrange, and renamed the subject the calculus of variations.

In the summary to his first paper using variations, Euler says "Even though the author of this [Euler] *had meditated a long time and had revealed to friends his desire yet the glory of first discovery was reserved to the very penetrating geometer of Turin LA GRANGE, who having used analysis alone, has clearly attained the same solution which the author had deduced by geometrical considerations."*

Lagrange derived the necessary condition

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}, \tag{10}$$

known today as the "Euler-Lagrange equation." (This was not his notation. The symbol ∂ for partial derivative was first used by Legendre in 1786.)

Equation (10) makes perfect sense and is a necessary condition for optimality for a vector-valued variable q as well as for a scalar one. It can be written as a system:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} = \frac{\partial L}{\partial q^i}, \quad i = 1, \ldots, n. \tag{11}$$

Alternatively, we can regard Equation (10) as a vector identity, in which $q = (q^1, \ldots, q^n)$ is an n-dimensional vector, and $\frac{\partial L}{\partial q}, \frac{\partial L}{\partial \dot{q}}$ stand for the n-tuples $\left(\frac{\partial L}{\partial q^1}, \ldots, \frac{\partial L}{\partial q^n}\right), \left(\frac{\partial L}{\partial \dot{q}^1}, \ldots, \frac{\partial L}{\partial \dot{q}^n}\right)$. A modern mathematician might be troubled by the use of $\dot{q}$ both as an "independent variable" and as a function of time evaluated along a trajectory, and might prefer to write (10) as

$$\frac{d}{dt}\left[\frac{\partial L}{\partial u}\big(q(t), \dot{q}(t), t\big)\right] = \frac{\partial L}{\partial q}\big(q(t), \dot{q}(t), t\big), \quad i = 1, \ldots, n, \tag{12}$$

where the Lagrangian $L(q, u, t)$ is a function on $\mathbb{R}^{2n+1}$, i.e. a function of $q \in \mathbb{R}^n$, $u \in \mathbb{R}^n$, $t \in \mathbb{R}$. This makes it clear that to compute the left-hand side of (10) one first evaluates $\frac{\partial L}{\partial \dot{q}}$ "treating $\dot{q}$ as an independent variable," then plugs in $q(t)$ and $\dot{q}(t)$ for q, $\dot{q}$, and finally differentiates with respect to t.

The Euler-Lagrange system (10)—or (12)—only gave conditions for stationarity, i.e., for the first variation of I to be zero. The next natural step was to look at the second variation, and this was done by Legendre (1752-1833), who found an additional necessary condition for a minimum. His condition, derived for the scalar case, is

$$\frac{\partial^2 L}{\partial \dot{q}^2}\big(q(t), \dot{q}(t), t\big) \geq 0 \quad \left(\text{i.e., } \frac{\partial^2 L}{\partial u^2}\big(q(t), \dot{q}(t), t\big) \geq 0\right). \tag{13}$$

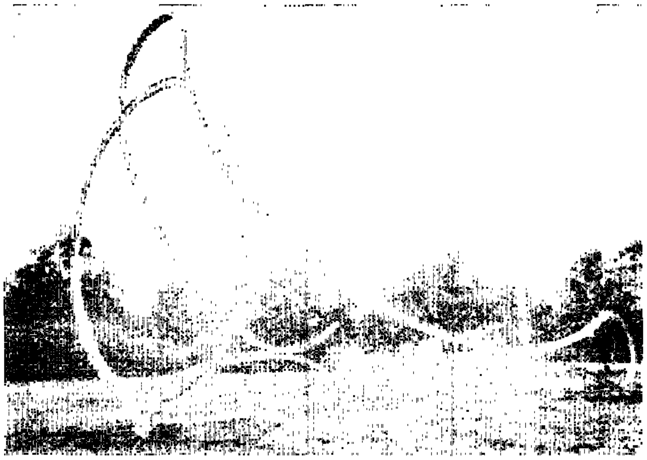

Fig. 6. The Brachystochrone Monument.

With an appropriate reinterpretation, Legendre's condition (13) is also necessary in the vector case: all we have to do is read (13) as asserting that the Hessian matrix $\left\{\frac{\partial^2 L}{\partial u^i \partial u^j}\left(q(t), \dot{q}(t), t\right)\right\}_{1 \le i,j \le n}$ has to be nonnegative definite.

The First Fork in the Road: Hamilton

At this point, we are close to the first and most critical fork in the road, involving the work of W.R. Hamilton (1805-1865). In a sense, the issue at stake will seem rather trivial, just a matter of rewriting the Euler-Lagrange system in a different formalism. However, sometimes formalisms can make a tremendous difference. To understand what happened and what could have happened but did not, let us try to make sense of the two necessary conditions for a minimum that have been presented so far. We have the *Euler-Lagrange equation* (10) and the *Legendre condition* (13). The Legendre condition is clearly the second-order necessary condition for a minimum of a function, namely, $L(q(t), u, t)$ as a function of u, but (10) does not look at all like the first-order condition for a minimum of that same function. It is natural to ask whether there might be a way to relate the two conditions. Is it possible that both can be expressed as necessary conditions for a minimum of one and the same function? The answer is yes, and understanding how this is done leads straight to optimal control theory, the maximum principle, and far-reaching generalizations of the classical theory. But before we get there, let us tell the story of how Hamilton almost got there himself, but missed, and Weierstrass got even closer, but missed as well.

Let us look at another way of writing (10). Suppose a curve $t \mapsto q(t)$ is a solution of (10). Define a function $H(q, u, p, t)$ of three vector variables q, u, p in $\mathbb{R}^n$, and of $t \in \mathbb{R}$, by letting

$$H(q, u, p, t) = <p, u> - L(q, u, t) \qquad (14)$$

Then define

$$p(t) = \frac{\partial L}{\partial u}\left(q(t), \dot{q}(t), t\right). \qquad (15)$$

It is then clear that $\frac{\partial H}{\partial p} = u$, so along our curve $q(t)$:

$$\frac{dq}{dt}(t) = \frac{\partial H}{\partial p}\left(q(t), \dot{q}(t), p(t), t\right). \qquad (16)$$

Also, $\frac{\partial H}{\partial q} = -\frac{\partial L}{\partial q}$, so (12), with $p(t)$ defined by (15), says that

$$\frac{dp}{dt}(t) = -\frac{\partial H}{\partial q}\left(q(t), \dot{q}(t), p(t), t\right). \qquad (17)$$

Finally, $\frac{\partial H}{\partial u} = p - \frac{\partial L}{\partial u}$, so (15) says:

$$\frac{\partial H}{\partial u}\left(q(t), \dot{q}(t), p(t), t\right) = 0. \qquad (18)$$

The system of equations (16), (17), (18), usually written more concisely as

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q}, \quad \frac{\partial H}{\partial u} = 0, \qquad (19)$$

is exactly equivalent to (10), provided that H is defined as in (14).

We will call the function H the "control Hamiltonian," and refer to (19) as the *control Hamiltonian form* of the Euler-Lagrange equations. In our view, Formula (14) is the definition that Hamilton should have given for the Hamiltonian, and Equations (19) are *"Hamilton's equations as he should have written them."*

What Hamilton actually wrote was (in our notation, not his)

$$\frac{dq}{dt} = \frac{\partial \mathcal{H}}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial \mathcal{H}}{\partial q}, \qquad (20)$$

where $\mathcal{H}(q, p, t)$ is a function of p,q and t alone, defined by the formula $\mathcal{H}(q, p, t) = \langle p, \dot{q}\rangle - L(q, \dot{q}, t)$, which resembles (14), but is not at all the same. The difference is that in Hamilton's definition, $\dot{q}$ is supposed to be treated *not* as an independent variable, but as a function of q, p, t, defined implicitly by the equation

$$p = \frac{\partial L}{\partial \dot{q}}(q, \dot{q}, t). \qquad (21)$$

It is easy to see that, *if the map* $(q, \dot{q}, t) \to (q, p, t)$ *defined by (21) can be inverted*, i.e., if we can "solve (21) for $\dot{q}$ as a function of q, p, t," then (20) is equivalent to (19). Indeed, it is clear that $\mathcal{H}(q, p, t) = H(q, u(q, p, t), t)$, where $u = u(q, p, t)$ satisfies $\frac{\partial L}{\partial u}(q, u, t)$. So

$$\frac{\partial \mathcal{H}}{\partial q} = \frac{\partial H}{\partial q} + \frac{\partial H}{\partial u} \cdot \frac{\partial u}{\partial q}. \qquad (22)$$

Since $\frac{\partial H}{\partial u}(q, u, t) = 0$ for $u = u(q, p, t)$, we see that $\frac{\partial \mathcal{H}}{\partial q} = \frac{\partial H}{\partial q}$ along solutions of (19), and then the first equation of (20) holds as well. Similarly, the second equation of (20) also holds. The converse is also easily proved.

It should be clear from the above discussion that the Hamiltonian reformulation of the Euler-Lagrange equations in terms of the "control Hamiltonian" is at least as natural as the classical one, and perhaps even simpler. Moreover, the control formulation has at least one obvious advantage, namely,

(A1) *the control version of the Hamilton equations is equivalent to the Euler-Lagrange system under completely general conditions, whereas the classical version only makes sense when the transformation (21) can be inverted, at least locally, to solve for* $\dot{q}$ *as a function of* q, p, t.

We now show that (A1) is not the only advantage of the control view over the classical one. To see this, we must take another look at Legendre's condition (13). Since $H(q, u, p, t)$ is equal to $-L(q, u, t)$ plus a linear function of u, (13) is completely equivalent to

$$\frac{\partial^2 H}{\partial u^2}(q(t), \dot{q}(t), p(t), t) \le 0,$$

$$\text{i.e., } \frac{\partial^2 H}{\partial \dot{q}^2}(q(t), \dot{q}(t), p(t), t) \le 0. \tag{23}$$

Now let us write (23) side by side with the third equation of (19):

$$\frac{\partial H}{\partial u} = 0 \quad \text{and} \quad \frac{\partial^2 H}{\partial u^2} \le 0. \tag{24}$$

and let us stare at the result for a few seconds.

These equations unmistakably suggest something! Clearly, what *has* to be going on here is that H must have a maximum as a function of u. So we state this as a conjecture.

CONJECTURE M: besides (19) (or the equivalent form (10)), an additional necessary condition for optimality is that $H(q(t), u, p(t), t)$, as a function of u, have a maximum at $\dot{q}(t)$ for each t.

Notice that Conjecture M is a natural consequence of rewriting Hamilton's equations "as Hamilton should have done it," and it is reasonable to guess that, if Hamilton had actually done it, then he himself, or some other 19th century mathematician, would have written (24) and be led by it to the conjecture. On the other hand, it is only by using the Hamiltonian of (14), as opposed to Hamilton's own form of the Hamiltonian, that one can see that *the Legendre condition has to do with the sign of the second u-derivative of a function of u whose first u-derivative has to vanish.* This function cannot be L itself, because the first order conditions do *not* say that $\frac{\partial L}{\partial u} = 0$. Nor can it be Hamilton's Hamiltonian $\mathcal{H}$, which isn't even a function of u. *Only the use of the "control" Hamiltonian leads naturally to Conjecture M.*

It turns out that Conjecture M is true, and that once its truth is known then vast generalizations are possible. But before we get there, we must move to the next chapter in our tale, and discuss the work of Weierstrass, who essentially discovered and proved Conjecture M, but did it using a language that obscured the simplicity of the result, and for that reason missed some profound implications of his discovery.

The Second Fork in the Road: Weierstrass

Weierstrass (1815-1897) considered the problem of minimizing an integral I of the form $I = \int_a^b L(q(s), \dot{q}(s))ds$ for Lagrangians L such that $L(q, \dot{q})$ *is positively homogeneous with respect to the velocity $\dot{q}$ (that is, $L(q, \alpha\dot{q}) = \alpha L(q, \dot{q})$ for all q, $\dot{q}$ and all $\alpha \ge 0$) and does not depend on time.* (As will become clear soon, we have a good reason for using s rather than t as the "time" variable in the expression for I.)

In a sense, one can always make this assumption on L "without loss of generality," by defining a new function $\Lambda(q, t, u, \tau) = \tau L(q, u/\tau, t)$, and think of t as a new q variable, say q^0, and of τ as $\frac{dq^0}{ds}$, where s is a new time variable, or "pseudotime," not to be confused with the true time variable t. However, "without loss of generality" is a dangerous phrase, and does not at all entail "without loss of insight." We shall argue below that this restriction, in conjunction with the dominant view that Hamilton's equations had to be written in the form (20), may have served to conceal from Weierstrass the true meaning and the far-reaching implications of the new condition he discovered.

Weierstrass introduced the "excess function"

$$\varepsilon(q, u, \bar{u}) = L(q, \bar{u}) - \frac{\partial L}{\partial u}(q, u) \cdot \bar{u}, \tag{25}$$

depending on three sets of independent variables q, u, and $\bar{u}$. He then proved his *side condition*: *For a curve $s \mapsto q_*(s)$ to be a solution of the minimization problem, the function ε has to be ≥ 0 when evaluated for $q = q_*(s)$, $u = \dot{q}_*(s)$, and a completely arbitrary $\bar{u}$.*

Weierstrass derived this side condition by comparing the reference curve q_* with other curves $q(\cdot)$ that are "small perturbations" of q_*, in the sense that $q(s)$ is close to $q_*(s)$ for all s but $\dot{q}(s)$ need not be close to $\dot{q}_*(s)$. Since Weierstrass' condition involves comparing $L(q_*(s), u)$ for u close to $\dot{q}_*(s)$, with $L(q_*(s), u)$ near an arbitrary value $\bar{u}$ of u, possibly very far from $\dot{q}_*(s)$, it is obvious that variations q with "large" values of $\dot{q}$ are needed.

Notice that, for Lagrangians with the homogeneity property of Weierstrass, $L(q, u) = \frac{\partial L}{\partial u}(q, u) \cdot u$, so Weierstrass could equally well have written his excess function as

$$\varepsilon(q, u, \bar{u}) = L(q, \bar{u}) - \frac{\partial L}{\partial u}(q, u) \cdot \bar{u} - \left(L(q, u) - \frac{\partial L}{\partial u}(q, u) \cdot u\right). \tag{26}$$

Using $p = \frac{\partial L}{\partial u}(q, u)$ as in (15), we see that

$$\varepsilon(q, u, \bar{u}) = \left(L(q, \bar{u}) - \langle p, \bar{u}\rangle\right) - \left(L(q, u) - \langle p, u\rangle\right), \tag{27}$$

which the reader will immediately recognize as

$$\varepsilon(q, u, \bar{u}) = H(q, u, p) - H(q, \bar{u}, p), \tag{28}$$

where H is our "control Hamiltonian." So Weierstrass' condition, expressed in terms of the control Hamiltonian, simply says that **(MAX)** *along an optimal curve $t \mapsto q_*(t)$, if we define $p(t)$ via (15), then for every t, the value $u = \dot{q}_*(t)$ must maximize the (control) Hamiltonian $H(q_*(t), u, p(t), t)$ as a function of u.*

In Weierstrass' formulation, the condition was stated in terms of the excess function, for the special Lagrangians satisfying his homogeneity assumption. In that case the resulting H is independent of time, as in our equation (28). But, if one rewrites Weierstrass's condition as we have done, in terms of H, then one can take a general Lagrangian, transform the minimization problem into one in Weierstrass's form, write the Weierstrass condition in the form (MAX) (so in particular H is independent of time) and then undo the transformation and go back to the original problem. The result is (MAX), as written, with the control Hamiltonian of the original problem. So the Weierstrass condition, if reformulated as in (MAX), is valid for *all* problems, with exactly the same statement.

Moreover, (MAX) can be simplified considerably. Indeed, the requirement that $p(t)$ be defined via (15) is now redundant: if $H(q(t), u, p(t), t)$, regarded as a function of u, has a maximum at

$u = \dot{q}(t)$, then $\frac{\partial H}{\partial u}\big(q(t), \dot{q}(t), p(t), t\big)$ has to vanish, so $p(t)$ has to be given by (15). Moreover, the vanishing of $\frac{\partial H}{\partial u}\big(q(t), \dot{q}(t), p(t), t\big)$ is also one of the conditions of (19). So we can state (19) and (MAX) together:

(NCO) *If a curve $t \mapsto q(t)$ is a solution of the minimization problem (1), then there has to exist a function $t \mapsto p(t)$ such that the following three conditions hold for all t:*

$$\dot{q}(t) = \frac{\partial H}{\partial p}\big(q(t), \dot{q}(t), p(t), t\big),$$

$$\dot{p}(t) = -\frac{\partial H}{\partial q}\big(q(t), \dot{q}(t), p(t), t\big), \tag{29}$$

$$H\big(q(t), \dot{q}(t), p(t), t\big) = \max_u H\big(q(t), u, p(t), t\big). \tag{30}$$

As a version of the necessary conditions for optimality, (NCO) encapsulates in one single statement the combined power of the Euler-Lagrange necessary conditions and the Weierstrass side condition as well, of course, as the Legendre condition, which obviously follows from (MAX). Notice the elegance and economy of language achieved by this unified statement: there is no need to bring in an extra entity called the "excess function." Nor does one need to include a formula specifying how $p(t)$ is defined, since (30) does this automatically. So the addition of the new Weierstrass condition to the three equations of (19) results in a new set of three, rather than four, conditions, a set much "simpler than the sum of its parts." Notice moreover that (MAX)—or, more precisely, the Weierstrass side condition part of (MAX)—is exactly Conjecture M. So we can surmise at this point that (MAX), as stated, probably could have been discovered soon after the work of Hamilton, since it is strongly suggested by (24), and almost certainly by Weierstrass, if only Hamilton's equations had been written in the form (14), (19).

So, we can now add two new items to our list of advantages of the "control formulation" of Hamilton's equations over the classical one:

(A2) *Using the control Hamiltonian, it would have been an obvious next step to write Legendre's condition in "Hamiltonian form," as in (24), and this would have led immediately to the formulation of Conjecture M, a proof of which would then have been found soon after.*

(A3) *With the control Hamiltonian, Weierstrass's side condition becomes much simpler, does not require the introduction of an "excess function," and can be combined with the Hamilton equations into an elegant unified formulation (NCO) of the necessary conditions for optimality.*

But this is by no means the end of our story. There is much more to the new formulation (NCO) than just elegance and simplicity. If you compare (NCO) with all the other necessary conditions that we had written earlier, a remarkable new fact becomes apparent. Quite amazingly, *the derivatives with respect to the u variable are gone.* All the earlier equations involved u-derivatives of L or of H, and even if we use the classical version (20) of Hamilton's equations, which involves no functions of u and therefore no u-derivatives, the fact remains that in order to get to (20), we first have to solve (21), which does involve a u-derivative.

Now, if our necessary conditions for optimality can be stated without any references to u-derivatives, we can apply the well-known *Principle of Mathematical Guessing*[6], which in the case at hand suggests that the existence of the u-derivative of L is not needed. Then there is no longer any reason to insist that the range of values of u be the whole space: any subset of $\mathbb{R}^n$ would do, since the minimization that occurs in (30) makes sense over any set. This leads us to

CONJECTURE M2: (NCO) should still be a necessary condition for optimality even for problems where $\dot{q}$ is restricted to belong to some subset U of $\mathbb{R}^n$, and $L(q, u, t)$ is not required to be differentiable with respect to u.

Now that we have liberated ourselves from the constraint that L be differentiable with respect to u, it ought to be possible for u—i.e., $\dot{q}$—to be anything, and (NCO) will still work. Once this is understood, the next natural step is to apply the Principle of Mathematical Guessing once again and allow $\dot{q}$ to be even "more arbitrary," for example a general function of some other variable u, and of q and t. So, instead of letting $\dot{q}$ be u, we can write $\dot{q} = f(q, u, t)$ for a general function $f(q, u, t)$. In that case, the expression $<p, u>$—i.e. $<p, \dot{q}>$—that occurs in (14) should of course be replaced by $<p, f(q, u, t)>$. This leads us to

CONJECTURE M3: (NCO) should still be a necessary condition for optimality even for problems where q is restricted to satisfy a differential equation $\dot{q} = f(q, u, t)$, with the "control function" $t \mapsto u(t)$ taking values in some set U and allowed to be a "completely arbitrary" U-valued function of t, and the Hamiltonian H now being defined by

$$H(q, u, p, t) = <p, f(q, u, t)> - L(q, u, t) \tag{31}$$

Those readers who are familiar with optimal control theory will, of course, have recognized Conjecture M3 as being essentially the same thing as the celebrated "Pontryagin maximum principle."

And we hope to have convinced *all* readers, even those who are not control theorists, that (NCO) is a very natural conclusion. It should be clear from our discussion that (NCO) could have been guessed almost immediately from "Hamilton's equations as Hamilton should have written them," together with the Legendre condition, and would have been an almost obvious conjecture to make once the Weierstrass side condition is known, *if only the "correct" Hamiltonian formalism, as in (14) and (19), had been used all along.*

The Maximum Principle

So far, we have shown that Conjecture M3 is almost forced on us if one looks at the classical condition from the right perspective and with the right formalism, but we have not yet said whether it is actually true, nor have we given any indication as to how one might go about proving it.

It turns out, however, that Conjecture M3, as stated, is not true, as can be seen from simple examples, but only a minor modification is needed to make it true. All we have to do is introduce a new p-variable p_0—the "abnormal multiplier"[7]—and write the Hamiltonian as

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[6] If a statement is proved under some specific restrictions but turns out not to involve these restrictions at all, chances are that the restrictions are not needed and the statement is valid even without them.

$$H(q, u, p, p_0, t) = <p, f(q, u, t)> - p_0 L(q, u, t) \qquad (32)$$

Everything we have done until now corresponded to taking $p_0 = 1$. We now impose, instead, the weaker requirement that $\dot{p}_0 = 0$ (i.e., p_0 is a constant) and $p_0 \geq 0$. We then observe that, if we use this new H rather than the old one, then the three conditions of (NCO) are always satisfied if we make the trivial choice $p(t) \equiv 0, p_0 = 0$. So, our new conditions will give nothing interesting unless we impose a further *nontriviality condition*, stating that this possibility is excluded.

With Conjecture M3 adjusted with the introduction of the abnormal multiplier p_0, we have finally reached the justly celebrated *Maximum Principle*:

(MP) *For the problem of minimizing a functional* $I = \int_a^b L(q(t), u(t), t)dt$ *subject to a dynamical constraint (3), and endpoint constraints* $q(a) = \bar{q}$, $q(b) = \hat{q}$, *with the parameter* u *belonging to a set* U, *the variable* q *taking values in* $\mathbb{R}^n$*—or in a open subset* Q *of* $\mathbb{R}^n$*—and the time interval* $[a,b]$ *fixed, a necessary condition for a function* $t \mapsto u_*(t)$ *on* $[a, b]$ *and a corresponding solution* $t \mapsto q_*(t)$ *of (3) to solve the minimization problem is that there exist a function* $t \mapsto p_*(t) \in \mathbb{R}^n$ *and a constant* $p_0 \geq 0$ *such that*

(*NT*) $(p_*(t), p_0) \neq (0,0)$ *for all* $t \in [a, b]$;

(*HS*)] $\dot{q}_*(t) = \frac{\partial H}{\partial p}(\Xi_*(t))$ *and* $-\dot{p}_*(t) = \frac{\partial H}{\partial q}(\Xi_*(t))$ *for* $t \in [a,b]$;

(*MC*) $H(\Xi_*(t)) = \max_{u \in U} H(q_*(t), u, p_*(t), p_0, t)$ *for* $t \in [a,b]$,

where we have written $\Xi_*(t) = (q_*(t), u_*(t), p_*(t), p_0, t)$, *and the Hamiltonian* $H(q, u, p, p_0, t)$ *is given by* (32).

Conditions (NT), (HS), and (MC) are known, respectively, as the *nontriviality condition*, the *Hamiltonian system*, and the *minimization condition*. Notice that (HS) is just a restatement of (29), with the new H, and (MC) is a restatement of (30). The second equation of (HS) is called the *adjoint equation*. A trajectory-control pair (q_*, u_*) for which there exist p_*, p_0 with the properties of (MP) is called an *extremal*.

Finally, we remark that for classical calculus of variations problems (MP) yields exactly the same conclusion as (NCO). Indeed, in this case it is possible to exclude the possibility that $p_0 = 0$, and (MP) reduces to (NCO). So (MP), as stated, is a true generalization of the necessary conditions (NCO), which covers many cases that cannot be handled by means of the classical calculus of variations.

We conclude by presenting the analogue of (MP) for problems with a variable time interval:

(MP') *For a minimization problem of the kind discussed in (MP), but with the time interval* $[a, b]$ *not fixed in advance, assuming that* f *and* L *do not depend on* t, *the necessary conditions are exactly the same as those of (MP), plus the extra requirement that* $H(q_*(t), u_*(t), p_*(t), p_0) \equiv 0$.

Statement (MP') applies in particular to *minimum time problems*, i.e., problems where $L \equiv 1$.

From Principle to Theorem

Our discussion so far has dealt only with the formal aspect of the necessary conditions for optimality. In order to get real mathematical theorems, we have to be accurate as to the technical assumptions on L, f, and U, the exact statement of the problem, and the precise meaning of the conclusions.

The results of the previous sections, from the Euler-Lagrange equation to the maximum principle, should be regarded as *principles* rather than theorems. For us, a *principle* is a generator of theorems, a not yet completely precise statement that can be made into a theorem by filling in the technical details and making all the definitions and conditions completely precise. The resulting theorems are *versions* of the principle. Usually, the choice of technical conditions can be made in more than one way, so a "principle" has more than one version.

In some cases, a "principle" becomes identified in the minds of mathematicians with its *first* published rigorous version. This has happened to some extent in the case of the maximum principle, because the book [8], where the result was first presented, already contains a rigorous version. We contend, however, that this version does not exhaust the full power of the principle, and the work of stating and proving stronger and more general versions is still very much in progress.

Regarding the necessary conditions for optimality, while the discovery of new and more general formal conditions progressed, rigorous versions of the formal results were derived at various stages of the process, using in each case the mathematical tools available at the time.

The first rigorous version of the maximum principle appears in the book [8]. This "classical" version was then improved by other authors. We choose to quote a version appearing in L.D. Berkovitz's 1974 book [2].

"Let $f^1, ..., f^n$ be the components of f, and write f^0 for L. It is assumed that the f^i, for $i = 0, ..., m$, are defined on Q x U_0 x $[a, b]$, where Q, U_0 are open subsets of $\mathbb{R}^n$, $\mathbb{R}^m$, respectively. Moreover, each function $q \mapsto f^i(q, u, t)$ is required to be of class C^1 with respect to q for each $(u, t) \in U_0$ x $[a, b]$, and each map $(u, t) \mapsto f^i(q, u, t)$ has to be Borel measurable for each fixed $q \in Q$. The set U is a subset of U_0. An *admissible control* is a map $[a, b] \mapsto u(t) \in U$ such that for every compact subset K of Q there is an integrable function $t \mapsto \varphi_K(t)$ such that the bound $\|f^i(q, u(t), t)\| + \left\|\frac{\partial f^i}{\partial q}(q, u(t), t)\right\| \leq \varphi_K(t)$ holds for all $(q, t) \in K$ x $[a, b]$ and all $i = 0, ..., m$. For a general class $\mathcal{U}$ of U-valued functions on $[a, b]$, and $\bar{q}, \hat{q} \in Q$, let us use $\mathcal{C}(\mathcal{U}, \bar{q}, \hat{q})$ to denote the set of all pairs $(q(\cdot), u(\cdot))$ such that $u(\cdot) \in \mathcal{U}$, $q(\cdot)$ is a solution of (3) (i.e., $q(\cdot)$ is an absolutely continuous curve $[a, b] \mapsto Q$ such that (3) holds for almost every t), $q(a) = \bar{q}$, and $q(b) = \hat{q}$. Use a $\mathcal{U}_{adm}$ to denote the class of all admissible controls. Then the optimization problem is that of minimizing the integral $I = \int_a^b L(q(t), u(t), t)dt$ in the class $\mathcal{C}(\mathcal{U}_{adm}, \bar{q}, \hat{q})$. The conclusion of the theorem is that of (MP), with p_* absolutely continuous, and the adjoint equation and the maximization condition holding almost everywhere."

The proof of this first version of the maximum principle is rather long, and we will not even sketch it here. Since then, stronger versions have been obtained by weakening the hypothesis of the first version, or strengthening the conclusions, or both.

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[7] The need for the abnormal multiplier had already been noticed by Bolza in 1913, cf. [3].

One important improvement of the classical version resulted from the use of *nonsmooth analysis* (cf. Clarke [4, 5]). While these "nonsmooth" generalizations were being developed, other authors pursued a different direction, for very smooth systems. They observed that one could get stronger results by allowing a class of variations richer than that used in the classical proof. One can then obtain "high-order necessary conditions for optimality." In addition, a third direction developed in which (MP) is formulated not for controlled differential equations $\dot{q} = f(q, u, t)$, but for *differential inclusions* $\dot{q} \in F(q, t)$, where F is a set-valued map (cf. for example [5]). The results referred to are proved by different methods and cannot be combined into a single theorem. We will not attempt to explain why this is so, because to do it we would have to discuss in detail the proofs of these theorems, showing that in each case one uses a different construction, and these constructions cannot be combined into a single one valid on the whole interval. But it is a fact that, due to this incompatibility of the various proofs, a single theorem covering all cases and combining them—that is, applying to "hybrid" problems as above—appeared, until a few years ago, to be beyond reach. Recently, however, one of us (Sussmann [10-12]) has obtained a general version of (MP) that contains all the above results, applies to some new cases as well, and actually covers the "hybrid" case.

Finale for Brachystochrone and Control

We conclude by returning to the brachystochrone problem, this time from the perspective of optimal control theory.

We can formulate Bernoulli's question as an optimal control problem in the x,y plane, whose dynamics are given by

$$\dot{x} = u\sqrt{|y|}, \quad \dot{y} = v\sqrt{|y|}\,, \tag{33}$$

where the control is a 2-dimensional vector (u,v) taking values in the set $U = (u, v) : u^2 + v^2 = 1$.

The Hamiltonian $H(x, y, u, v, p, q, p_0, t)$ is then given (using $\alpha = \operatorname{sgn} y$) by the formula $H = (p_1 u + p_2 v)\sqrt{\alpha y} - p_0$, and the application of (NCO) gives the conditions

$$u = \frac{p_1}{|p|}\,, \quad v = \frac{p_2}{|p|}\,, \tag{34}$$

where $|p| = \sqrt{p_1^2 + p_2^2}$, as well as the differential equations

$$\dot{p}_1 = 0\,, \quad \dot{p}_2 = -\alpha\frac{p_1 u + p_2 v}{2\sqrt{\alpha y}} = -\frac{\alpha|p|}{2\sqrt{\alpha y}}\,. \tag{35}$$

Notice that $|p(t)| \neq 0$. Indeed, (MP') tells us that $H = 0$. So $|p| = 0$ would imply $p_0 = 0$, contradicting (NT).)

If the constant p_1 vanishes, then $\dot{x} \equiv 0$, so we get a vertical line. Otherwise, $\dot{x}$ is continuous and always $\neq 0$, *showing that we can use x to parametrize our solution.* Since $y'(x) = \frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{v}{u} = \frac{p_2}{p_1}$, we have $1 + y'(x)^2 = \frac{|p|^2}{p_1^2}$ and

$$y''(x) = \frac{1}{p_1}\cdot\frac{dp_2}{dx} = \frac{\dot{p}_2}{p_1\dot{x}}\,. \tag{36}$$

But (33) and (34) imply that $\dot{x} = \frac{p_1\sqrt{\alpha y}}{|p|}$, and then Equations (35) and (36) yield $y''(x) = -\frac{|p|^2}{2yp_1^2}$. So $2yy'' = -\frac{|p|^2}{p_1^2} = -(1 + y'^2)$, and then $1 + y'^2 + 2yy'' = 0$, which is exactly Equation (9). As we explained before, this leads to the cycloids, with no "spurious solutions." Notice that this argument does not involve any discretization or any use of refraction of light across boundaries.

Notice also that in our control argument *we have not postulated that the solution curves could be represented as graphs of functions y(x). We have proved it!* (In the calculus of variations case this was an extra assumption, cf. "Bernoulli's Solution of the Brachystochrone Problem" above.)

This is one example showing that, for the brachystochrone problem, *the optimal control method gives better results than the classical calculus of variations.*

All the above considerations apply to the computation of optimal trajectories that are entirely above the x axis, as in Bernoulli's brachystochrone problem. However, the natural mathematical setting for the minimum time control problem corresponding to (33) is the whole plane, which is why we wrote $\sqrt{|y|}$ rather that $\sqrt{y}$ in (33). It is natural, therefore, to try to solve this more general problem, i.e., to try to find the light rays when the medium is *the whole plane*, and the speed of light is $\sqrt{|y|}$. Notice that this problem is "completely controllable," in the sense that any two points A, B of $\mathbb{R}^2$, even if they lie on opposite sides of the x axis, can be joined by a feasible path. The right-hand side of (33) vanishes along the x axis, but this does not prevent the existence of feasible paths crossing the x-axis, because the function $\sqrt{|y|}$ is not Lipschitz near the x axis. (If the function was Lipschitz, then by the usual uniqueness theorem of ordinary differential equations, every solution going through a point on the x axis would have to be a constant curve.) However the same non-Lipschitz feature that makes the system controllable also renders the maximum principle inapplicable, in its classical and nonsmooth versions, including the Lojasiewicz version, since all these require a Lipschitz reference vector field.

Suppose, for example, that we want to find an optimal trajectory from A to B, where A lies in the upper half-plane and B is in the lower half-plane. Then one can show, first of all, that an optimal trajectory ξ exists, using Ascoli's theorem. Next, using the usual necessary conditions for optimality, e.g., the Euler-Lagrange equation or the classical version of the maximum principle, one shows that any portion of an optimal curve which is entirely contained in the closed upper half plane or in the closed lower half plane is a cycloid given by (5), or a reflection of such a cycloid with respect to the x axis. Next, one sees that ξ cannot traverse the x axis more than once. (This requires an elementary qualitative lemma that we leave as exercise.) So we know that ξ consists of a cycloid going from A to a point X in the x axis, followed by a reflected cycloid going from X to B. It remains to find X.

It turns out that *the version of [12] applies*, since this result does not require Lipschitz continuity—or even continuity—of the right-hand side, and works as long as the reference trajectory arises from a semidifferentiable flow. We refer the reader to [12] for the details.

We conclude with one more example showing the *superiority of the optimal control method for the brachystrochrone problem*, by discussing the question of the rigorous proof of the optimality of Bernoulli's cycloids. Clearly, no argument based only on necessary conditions for optimality will ever prove that a trajectory is optimal. If we really want to prove the optimality of Bernoulli's cycloids, an extra step is needed. For example, it would suffice to prove *existence* of an optimal trajectory joining *A* and *B*. (Once this is established, it follows that the optimal trajectory is Bernoulli's cycloid, because this curve is the unique path joining *A* and *B* that satisfies the necessary conditions. The complete proof is a bit more complicated, because one needs an extra argument to exclude the possibility of cycloids that touch the x axis more than once before reaching *B*.) From the perspective of the classical calculus of variations, this is a hard problem,[8] because the Lagrangian given by (8) has a singularity at $y = 0$. In optimal control, however, the existence problem is trivial, since it suffices to apply Ascoli's theorem to the system (33) to obtain the desired result.

Would Bernoulli have liked this way of looking at his problem? Would he have appreciated the elegance with which optimal control can handle it? Would he have liked this approach better than the calculus of variations method? We let the reader be the judge.

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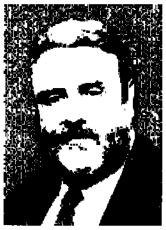

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[8] We thank F.H. Clarke for bringing this point to our attention.